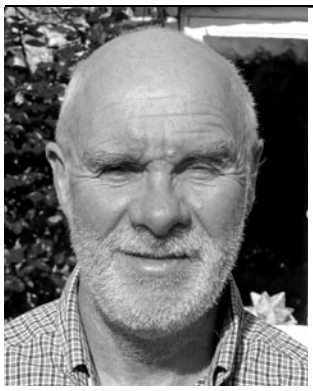
Chris Little

Differentiation in three easy, GeoGebra-style, lessons

Chris Little
University of Southampton
ctl@soton.ac.uk



Why is calculus difficult? It required a conjunction of geometric and algebraic thinking to invent or discover its concepts. What makes GeoGebra exciting as a tool for calculus is that it offers an opportunity to represent differentiation as both an algebraic and geometric process simultaneously. My experience of working with teachers is that the software is sufficiently straightforward and user-friendly that it doesn't take a lot of practice, or insider-knowledge about how it works, to give the teacher, or even the learner, the tools to discover the hidden treasures of calculus, and to develop insights which do not require the towering genius of its originators or discoverers, Newton and Leibnitz.

Take differentiation. First, we have the geometric image of a point moving round a curve, with its tangent varying in gradient (see Lesson 1 and Fig 1)

Lesson 1

1. Input f(x) = x^2
2. Create a **point** on f(x). Relabel (right click) the point as P.
3. **Construct** tangent to the curve at P.
4. **Measure** the slope (gradient) of the tangent (m).
5. **Move** P, and record the slope of the tangent at points with x-coordinates –2, –1, 0, 1, 2, …
6. Predict the gradient at the point with x-coordinate 4. Verify your prediction.
7. Predict the gradient at the point with x-coordinate 0.5. Verify your prediction.
8. Predict the gradient at the point with *x*-coordinate –2.5. Verify your prediction.
9. In your own words, write a rule for calculating the gradient of f(x) = x^2.
10. Repeat with f(x) = x^3, x^4, 2x^2, 2x^2 + 1, etc.

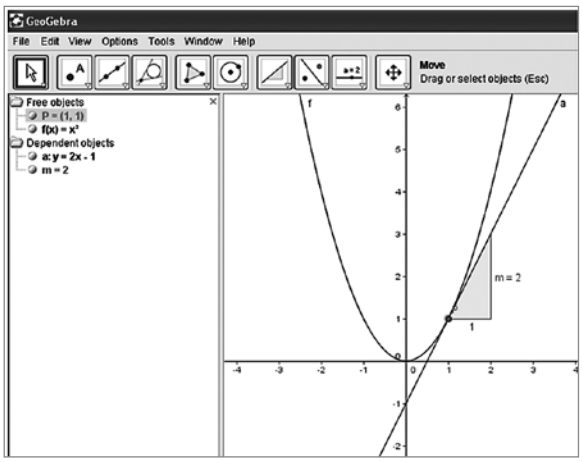


Fig 1 – Lesson 1 uses GeoGebra to create geometric image of tangent to curve varying in gradient

Second, we have the geometric image of a gradient function, traced by the value of the derivative as the point travels round the curve (see Lesson 2 and Fig 2)

Lesson 2

1. Input Q = (x(P), m) [x(P) is the *x*-coordinate of the point P].
2. **Move** P and observe what happens to the point Q.
3. Right click Q and turn trace on.
4. **Move** P, and observe the trace of Q. Find the equation of this trace line.
5. Undo trace (button top right)
6. **Construct** locus of Q (locus, then click on Q, then click on P). Find the equation of this locus line.
7. **Input** f′(x) = 2x. Verify that this is the same line as the locus line. Change colour (right click) to red.

f′(x) is called the *gradient function* of *derivative* of f(x). So the derivative of f(x) = x^2 is f′(x) = 2x.

8. What is the gradient of the tangent to f(x) = x^2 at (10, 100)?
9. What is the point on f(x) = x^2 where the gradient is 12?
10. What can you say about the gradient at $x = a$ and $x = -a$?
11. Repeat for f(x) = x^3, x^4, …, x^n…

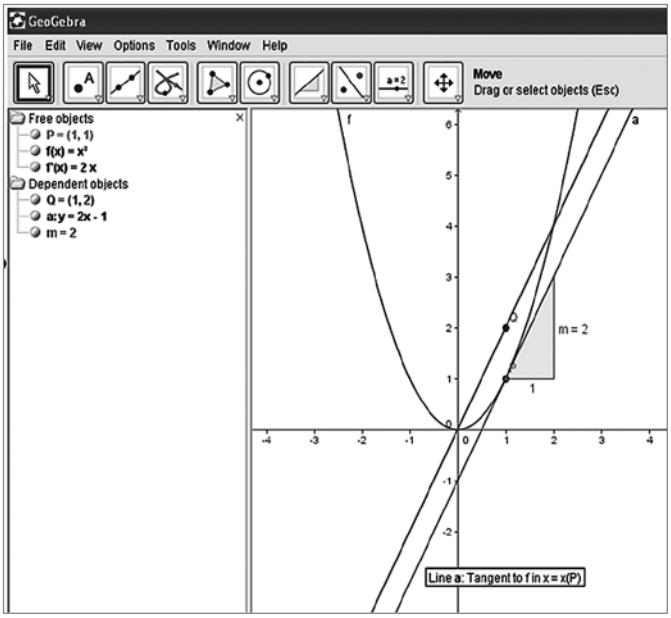


Fig 2 – Using GeoGebra to create geometric image a gradient function, traced by the derivative value as the point moves along the curve

Third, we have the algebraic process of differentiation, mirrored by its geometric image of the limit of the secant line as a tangent (see Lesson 3 and Fig 3).

All this could of course be done using other dynamic geometry packages, and Markus Hohenwarter would no doubt acknowledge the ground-breaking work of Cabri and Geometer's Sketchpad in pioneering computer interfaces which make all this possible! However, I believe that GeoGebra represents a significant advance, through its ease of use, uncluttered screen, and friendly on-screen help, which enables the user to focus on the maths rather than the mechanics of running the program.

Moreover, the closer match between the algebraic notation required for first principles differentiation, and the input of algebraic expressions in GeoGebra, might provide the learner with a vital, multiple–embodied, link between the geometric image (on the right) and algebraic process (on the left) which will help the process of encapsulation [1, 2] necessary to make sense of differentiation.

Lesson 3

1. **Input** f(x) = x^2 and create a **point** on the curve, labelling it P.
2. **Input** a = x(P). *Remember that a is the x-coordinate of P.*
3. **Construct** the tangent to the curve at P. Colour it red.
4. **Input** h=1. Right click and show object. In properties – slider – change min to 0, max to 1 and interval to 0.01. We are now going to create a point Q on the curve with *x*-coordinate x(P) + h.
5. **Input** Q = (a+h, f(a+h)) (be careful with the brackets!)
6. Change the value of h using the slider. Observe what happens to Q.
7. Create the **line** through PQ.
8. **Measure** the gradient (slope) of PQ (m).
9. **Move** the point P to (1, 1). Now gradually change the value of h until it approaches zero. Observe what happens to the chord PQ, and to the value of m. To get a closer look, you can zoom into P using the **move** menu. What happens to m when h = 0? Why? The gradient of the chord approaches (or *tends to*) the gradient of the tangent as h approaches (*tends to*) zero.
10. **Input** S = (a,m).
11. **Construct** the locus of S as P varies on the curve. Colour it red. This locus is the gradient of PQ plotted as a function of a, the *x*-coordinate of P.
12. What happens to this locus as h approaches zero? As h equals zero? We have so far used the slope function to find the gradient of PQ. Can we use algebra to do this? In the diagram (Fig 3) , PR = h, and QR = y-coord of Q – y-coord of P = f(a+ h) – f(a)

    So gradient of chord PQ = $\frac{QR}{PR} = \frac{f(a+h) - f(a)}{h}$

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13. **Input** m′=(f(a+h) – f(a))/h. Check that m = m′ as you change P or h.

    Now for f(x) = x^2, f(a) = a^2 and f(a+h) = $(a+h)^2$

    So the gradient of the chord

    $$PQ = m' = \frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - a^2}{h}$$

14. Expand $(a + h)^2$, and hence show that m′ = 2a + h (provided h is not zero!)

15. **Input** g(x) = 2x + h. Check that line is the locus of S. What happens to g(x) when h = 0?

    We say that the derivative f′(x) is the *limit* of $\frac{f(a+h) - f(a)}{h}$ as h tends to zero, or $\lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right]$

16. Repeat with f(x) = x^3, 2 x^2, etc.

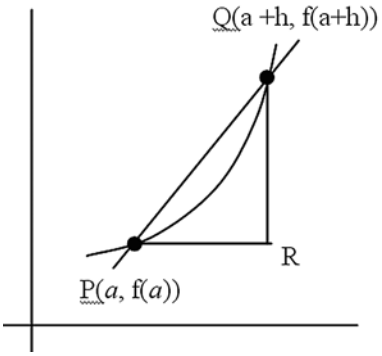

Fig 3 – Using GeoGebra to mirror the algebraic process of differentiation

References

1. Dubinsky, E. (1991) Reflective Abstraction in Advanced Mathematical Thinking. IN TALL, D. (Ed.) *Advanced Mathematical Thinking*. Dordrecht, Kluwer.
2. Sfard, A. (1991) On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics*, 22, 1-36.